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On a theory of amplitude vacillation in baroclinic waves

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This study contributes to the theory of amplitude vacillation for finite amplitude baroclinic waves in a two-layer, quasi-geostrophic, zonal flow as worked out by Pedlosky. In a recent paper the author has shown that Pedlosky omitted a certain side-wall boundary condition on the mean zonal flow. The neglect of this boundary condition results in an unspecified energy source at the side-wall boundaries and the physical problem is incorrectly posed.

In this paper, Pedlosky's analysis is repeated but with the side-wall boundary condition included. It is shown that the side-wall energy source is negligible only when the zonal wavenumber of the disturbance is large compared with the meridional wavenumber, and *not* otherwise. Moreover, the energy conversions to and from mean zonal kinetic energy corresponding to Pedlosky's calculations and those given here have essential differences, although for fixed meridional wavenumber, these differences become less pronounced as the zonal wavenumber increases.

It is also shown that, when the side-wall condition is included, the mean flow distortion associated with the wave is different in structure to that which occurs when the condition is omitted. However, as the total disturbance wavenumber a increases, the influence of the side wall on the mean flow structure is confined to a boundary layer of width comparable with the internal deformation radius $2^{\frac{1}{2}}a^{-1}$.

Even when the deformation radius is comparable with the channel width, the conclusions of Pedlosky (1972) concerning the existence of stable periodic solutions are correct, providing viscous effects are vanishingly small, and the criteria for stability of the steady solutions obtained herein are not significantly different from those given by Pedlosky. In this viscous regime, we have also studied the evolution to limit-cycle solutions.

Evolution in the case where viscous effects are small on the time scale for the initial growth of an incipient wave, but not vanishingly small, will be discussed in a subsequent paper.

1. Introduction

In a recent series of papers, Pedlosky (1970, 1971, 1972, henceforth referred to as P70, P71, P72) develops an analytic theory for the evolution to finite amplitude of marginally unstable baroclinic waves in a two-layer, quasi-geostrophic flow model and shows that different types of behaviour are possible according to the level of dissipation in the model. Thus P70 shows that, in the absence of frictional effects, the wave amplitude oscillates dnoidally as energy is extracted from and subsequently returned to the mean flow during an oscillation cycle. On the other hand, with dissipation present, a steady wave solution is possible in which the rate of energy transfer from the mean flow to the wave is exactly balanced by dissipation of wave energy. Moreover, if dissipative effects are significant on the time scale defined by the linear growth rate of the incipient wave, evolution to this steady solution always occurs. However, if dissipative effects are sufficiently small over this time scale, the steady wave solution is unstable for waves shorter than a certain wavelength, and according to P71, long-period pulsations in the wave amplitude evolve and are described by a stable limit-cycle solution of the wave amplitude equations. P72 gives analytic solutions for the limit cycles in the case of vanishingly small (but non-zero) dissipation and shows that solutions of this type are still possible even if the steady wave solution is stable, in which case an unstable limit cycle also exists. This presumably defines the range of attraction, in some sense, of the stable limit cycle and the steady solution.

In contrast to the inviscid dnoidal solution of P70, the limit-cycle solutions are independent of initial conditions and are therefore of interest as they may lead to an explanation for the occurrence of amplitude vacillation such as that observed in experiments with differentially heated rotating fluid annuli (see, for example, Pfeffer & Chiang 1967; Fowlis & Pfeffer 1969; Hide 1969) and in related experiments on mechanically driven baroclinic waves (Hart 1972, 1973), for certain values of the governing parameters. However, in a study of the energy conversion associated with Pedlosky's solutions for small dissipation, it is shown by the author (Smith 1974) that the omission of a certain boundary condition at the side boundaries of the flow results in an implicit, non-physical source of mean flow kinetic energy, the strength of which depends on the calculated amplitude of the wave.

In the present paper, the analysis of P72 is repeated with the side boundary conditions included and the uncertainty concerning the existence of limit cycles is resolved. Indeed, it is shown that the conclusions of P72 are broadly correct although there are major differences in detail between the two analyses. In general, the side-wall energy source in Pedlosky's calculations is not negligible compared with the internal conversion of mean energy to wave energy. Moreover, in the analysis of P72, the side-wall source is the primary one for maintaining the total energy level of the flow against the dissipation of wave energy.

The question as to whether or not an incipient wave evolves to a limit-cycle state is also studied here; this question was not addressed in P72.

Finally, we investigate the stability of the steady wave solution in the case of small, but not vanishingly small, dissipation. Some numerical integrations of the amplitude equations will be discussed in a second paper.

2. Reduction of the amplitude equations

In P71, a pair of coupled equations [cf. (4.9) and (4.10)] is derived for the growth of the wave amplitude $A(T)$ of a marginally unstable baroclinic wave and the distortion it produces in the original zonal mean flow, represented by its contribution $\Phi_1^{(2)}(y, T)$ to the total stream function, viz.

$$\frac{d^2A}{dT^2}+\frac{3}{2}\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{dA}{dT}-\frac{\Delta}{|\Delta|}\frac{k^2(U_1-U_2)^2}{4a^2}A +A\frac{k^2(U_1-U_2)}{2a^2}m\pi\int_0^1\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}\sin 2m\pi y\,dy=0, \quad (2.1)$$

$$\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}-a^2\Phi_1^{(2)}\right]+\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}=\frac{a^2m\pi}{2(U_1-U_2)}\left[\frac{d}{dT}|A|^2+\frac{2r}{|\Delta|^{\frac{1}{2}}}|A|^2\right]\sin 2m\pi y. \quad (2.2)$$

The notation is identical to that of P71. The domain of the equations is $0 \leqslant y \leqslant 1$ and $T \geqslant 0$ and the solution is completely determined when A, dA/dT and $\Phi_1^{(2)}(y, T)$ are specified at the initial instant $T = 0$ and the boundary conditions $\Phi_{1yT}^{(2)} = 0$ at $y = 0$ and $y = 1$ are satisfied for $T \geqslant 0$. *In the analyses of* P71 *and* P72, *the latter conditions are omitted and are tacitly replaced with the conditions* $\Phi_1^{(2)} = 0$ *at* $y = 0$ *and* $y = 1$ *by the choice*

$$\Phi_1^{(2)} = B(T)\sin 2m\pi y \quad (2.3)$$

as the form of the solution to (2.2).

At this stage we note that (2.1) and (2.2) have the steady solutions

$$\left.\begin{aligned} |A| = A_e &= (U_1-U_2)/am\pi, \\ \Phi_{1y}^{(2)} = (U_1-U_2)V_e(y) &= (U_1-U_2)(1-\cos 2m\pi y)/2m^2\pi^2, \end{aligned}\right\} \quad (2.4)$$

where for definiteness the boundary conditions $\Phi_{1y}^{(2)} = 0$ at $y = 0$ and $y = 1$ have been applied. This is equivalent to assuming that the wave grows from an incipient level and that there is no initial distortion of the mean flow, i.e. $\Phi_{1y}^{(2)}(y, 0) \equiv 0$. In this case, the above conditions follow immediately from the side-wall conditions $\Phi_{1yT}^{(2)} \equiv 0$ at $y = 0$ and $y = 1$.

The corresponding steady solution in Pedlosky's theory differs from (2.4) *only by the absence of the constant term in the expression for* $\Phi_{1y}^{(2)}$.

Before solving (2.1) and (2.2), it is convenient to rescale them by taking $A = A_e\tilde{A}$, $\Phi_1^{(2)} = (U_1-U_2)\Phi$, $T = 2a\tau/k(U_1-U_2)$ and $\sigma = 2ar/k(U_1-U_2)|\Delta|^{\frac{1}{2}}$, where τ is the rescaled time variable and σ is the rescaled viscous parameter. We also define $V = \Phi_y$ and use this as a dependent variable instead of Φ to circumvent the subsequent need to differentiate a Fourier series. Further, we assume henceforth that $\Delta > 0$ and, following P71, that A is real. Then (2.1) and the y derivative of (2.2) take the forms

$$\frac{d^2\tilde{A}}{d\tau^2}+\tfrac{3}{2}\sigma\frac{d\tilde{A}}{d\tau}-\tilde{A}\left[1+4m^2\pi^2\int_0^1 V\cos 2m\pi y\,dy\right]=0, \quad (2.5)$$

$$\frac{\partial}{\partial\tau}\left[\frac{\partial^2V}{\partial y^2}-a^2V\right]+\sigma\frac{\partial^2V}{\partial y^2}=2\tilde{A}\left(\frac{d\tilde{A}}{d\tau}+\sigma\tilde{A}\right)\cos 2m\pi y, \quad (2.6)$$

where the last term in (2.5) results from an integration by parts. When combined with the conditions $V(y,0) \equiv 0$, $V(0,\tau) = 0$ and $V(1,\tau) = 0$, this pair of equations is completely specified by prescribing initial values of $\tilde{A}$ and $d\tilde{A}/d\tau$.

In view of the expansion

$$\cos 2m\pi y = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n-1}{(2n-1)^2-4m^2}\sin(2n-1)\pi y, \quad 0 < y < 1,$$

we seek a solution of (2.6) in the series form

$$V(y,\tau) = \sum_{n=1}^{\infty} V_n(\tau)\sin(2n-1)\pi y \tag{2.7}$$

which satisfies the boundary conditions at $y = 0$ and $y = 1$ and the initial condition $V(y,0) \equiv 0$, provided that $V_n(0) = 0$ for each n. Then multiplication of (2.6) by $\sin(2n-1)\pi y$ and integration with respect to y from $y = 0$ to $y = 1$, using the boundary conditions on V, yields an ordinary differential equation for

$$V_n(\tau) = 2\int_0^1 V(y,\tau)\sin(2n-1)\pi y\, \mathrm{d}y.$$

Finally, with $X_n(\tau) = \frac{1}{8}\pi^3(2n-1)[4m^2-(2n-1)^2]V_n(\tau)$, the set of equations for the V_n, together with (2.5), reduces to the infinite autonomous system

$$\frac{d\tilde{A}}{d\tau} = B, \tag{2.8a}$$

$$\frac{dB}{d\tau} = \tfrac{3}{2}\sigma B + \tilde{A}\left[1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_n\right], \tag{2.8b}$$

$$\frac{dX_n}{d\tau} = q_n[\tilde{A}B + \sigma(\tilde{A}^2 - X_n)] \quad (n = 1, 2, \ldots), \tag{2.8c}$$

where

$$c_{mn} = [(2n-1)^2 - 4m^2]^{-2} \tag{2.9}$$

and

$$q_n = (2n-1)^2\pi^2/[(2n-1)^2\pi^2 + a^2]. \tag{2.10}$$

These equations have steady solutions $\tilde{A} = \pm 1$, $B = 0$, $X_n = 1$ $(n = 1, 2, \ldots)$, corresponding to (2.4); to see this note that

$$\sum_{n=1}^{\infty} c_{mn} = \pi^2/64m^2,$$

as shown in appendix A. Solutions which evolve with time may be obtained approximately by numerical integration.

At this stage it is apparent that the structure of the distorted mean zonal flow is, in general, quite different when the side-wall condition is included in the analysis; compare the forms for $\partial\Phi_1^{(2)}/\partial y$ obtained from (2.3) and (2.7), for example. Thus, in Pedlosky's analysis maximum distortion occurs at the side walls and for $m = 1$ there is none in mid-channel, whereas in the present theory there is no distortion at the side walls and a maximum in mid-channel, irrespective of the value of m. The differences may be seen most clearly in the case where $\sigma \ll 1$; then the solution is quasi-periodic and has the same spatial structure as the solution when $\sigma \equiv 0$ (see §§5.1 and 5.2). In the latter case, equation (4.11) of P71 gives

$$\Phi_{1y}^{(2)} \propto \cos 2m\pi y - \cosh a(y-\tfrac{1}{2})/\cosh \tfrac{1}{2}a \qquad (2.11)$$

and it can be shown that (2.7) leads to this structure also. Clearly $\Phi_{1y}^{(2)} = 0$ at $y = 0, 1$ and the side-wall boundary condition is satisfied. Moreover, the structure of (2.11) differs from that given in P72 (see the $\Phi_{1y}^{(2)}$ obtained from (2.3) above) by the term $\cosh a(y-\frac{1}{2})/\cosh \frac{1}{2}a$, which is unity at $y = 0, 1$ but decays exponentially away from each side wall with decay scale a^{-1}, or $2^{-\frac{1}{2}}$ times the internal radius of deformation. Hence for large m, and hence a, the effect of the side wall is experienced only in these side-wall layers.

3. Energy considerations

A comparison of the energy conversions in P71, P72 and the present theory shows considerable differences and the claim by Pedlosky (1975) that, in the parameter region of interest, the effect of the side-wall boundary term in his theory is energetically unimportant compared with the internal conversions is not wholly correct. According to the lowest-order energy analysis (see Smith 1974, figure 1), the ratio of side-wall energy input, say $\{SW, \bar{K}\}$, to internal potential energy conversion, $\{\bar{P}, P'\}$, is

$$-m\pi(U_1 - U_2)\, dB/dT \div \tfrac{1}{8}a^2(d|A|^2/dT + 2r|A|^2/|\Delta|^{\frac{1}{2}}).$$

In the case $0 < r/|\Delta|^{\frac{1}{2}} \ll 1$, studied in detail in P71, this ratio reduces to $4m^2\pi^2/(4m^2\pi^2 + a^2)$, using equations (6.1) and (4.9) in P71. For $m = 1$, limit cycles exist for values of a slightly less than 2π (see §5) and the side-wall energy conversion rate is only half the internal conversion rate in this case. However, when $a \gg 2m\pi$, corresponding to very short waves (horizontal wavenumber $k \gg 3^{\frac{1}{2}}m\pi$), the side-wall energy source is negligible compared with the internal conversion rate.

The ratio of side-wall energy input to the energy input $\{HB, \bar{K}\}$ from the horizontal boundaries is $|\Delta|^{\frac{1}{2}}\, dB/dT \div rB(T)$ and for $r/|\Delta|^{\frac{1}{2}} \sim 1$ this is typically of order unity unless the wave is steady. Furthermore, to order $|\Delta|^{\frac{3}{2}}$, the rate of change $d\bar{K}/dT$ of mean zonal kinetic energy is identically zero in Pedlosky's theory (Smith 1974, p. 2011). Hence the reservoir of this form of energy is unavailable for conversion to wave energy and could be regarded as a 'catalyst' for the conversion of side-wall energy, and that produced by the differential motion of the horizontal boundaries, into mean available potential energy and thence to wave energy, or vice versa. In the case $\sigma = 0$, $\{HB, \bar{K}\} \equiv 0$ and the conversion of mean zonal kinetic energy into mean available potential energy $\{\bar{K}, \bar{P}\}$, or vice versa, is associated entirely with a side-wall energy flux; i.e. $\{SW, \bar{K}\} \equiv \{\bar{K}, \bar{P}\}$. Hence the energy conversions corresponding to the solution given in P72 as $\sigma \to 0$ are *not* the same as those of the inviscid solution given in P70; in the latter, the side-wall energy transfer is zero and fluctuations in $\{\bar{K}, \bar{P}\}$ are associated with fluctuations in $\bar{K}$.

A similar situation emerges when $\sigma \ll 1$ but in this case there is an additional, but relatively small, replenishment of mean zonal kinetic energy, proportional to σ, from the horizontal boundaries. This maintains the total energy of the system against the slight viscous dissipation of wave energy. [N.B. If $\sigma \ll 1$, $r/|\Delta|^{\frac{1}{2}} \ll 1$ and from equation (6.2) in P71, $B(T) \sim |A|^2 \sim 1$. Hence from (3.1) above, $\{HB, \bar{K}\}/\{SW, \bar{K}\} \ll 1$.]

In contrast to these results, in the present theory $\{SW, \bar{K}\}$ is identically zero but $d\bar{K}/dT$ does *not* vanish identically to order $|\Delta|^{\frac{3}{2}}$. Thus, in the case $\sigma = 0$, $\{HB, \bar{K}\} \equiv 0$ and there is no dissipation of wave energy. Hence total energy (mean energy plus wave energy) is conserved and in particular, $\{\bar{K}, \bar{P}\}$ conversions are associated with changes in the level of $\bar{K}$, consistent with the energetics of the inviscid solution given in P70. For $\sigma \ll 1$, $\{HB, \bar{K}\}$ is non-zero but is again small compared with $d\bar{K}/dT$ and $\{\bar{K}, \bar{P}\}$, and serves to maintain the system against the slight dissipation of wave energy.

4. Steady solutions and their stability

The local stability of the steady solutions (2.4) to small perturbations of $\tilde{A}$ and V is most readily investigated by setting

$$\tilde{A} = \pm 1 + \alpha_0 e^{\lambda\tau} \quad \text{and} \quad V = V_e(y) + v_0(y)\, e^{\lambda\tau}$$

in (2.5) and (2.6) and seeking possible eigenvalues λ for which the linearized system of equations for α_0 and $v_0(y)$ has a solution, subject to the boundary conditions $v_0(y) = 0$ at $y = 0$ and $y = 1$. With $\lambda = \eta^2/(1-\eta^2)$, the problem reduces to finding the complex roots of the transcendental equation†

$$P(\eta) \equiv 4m^2\pi^2 + a^2\eta^2 - 4a\eta \tanh \tfrac{1}{2}a\eta + \frac{\sigma^2}{4m^2\pi^2}\frac{\eta^2(3-\eta^2)(4m^2\pi^2+a^2)^2}{(2-\eta^2)(1-\eta^2)^2} = 0. \quad (4.1)$$

Instability is assured if at least one eigenvalue λ has a positive real part. If $\eta = \eta_r + i\eta_i$, it is easy to show that roots of (4.1) giving values of λ with $\mathrm{Re}\,\lambda > 0$ lie in the small region S of the η_r^2, η_i^2 plane bounded by the line $\eta_i^2 = 0$ $(0 \leqslant \eta_r^2 \leqslant 1)$ and the parabola $\eta_r^2 - \eta_i^2 - (\eta_r^2 + \eta_i^2)^2 = 0$. The possibility of such roots is easily determined by plotting contours of $\mathrm{Re}\,(P(\eta)) = 0$ and $\mathrm{Im}\,(P(\eta)) = 0$ for values of η corresponding to points (η_r^2, η_i^2) in S (note that if $\eta = \eta_r + i\eta_i$ is a root of (4.1) then so also are the points $\pm\eta_r \pm i\eta_i$) and their values are readily obtained using Newton's method.

The regions of parameter space (i.e. the σ, a plane) in which the steady wave solution is stable or unstable are shown in figure 1 for the lower values of the transverse wavenumber m. It may be seen that for a given value of σ the longer waves are stable (smaller total wavenumber a) and the shorter ones unstable with the range of stable wavenumbers *decreasing* to a certain finite range (depending on m) as σ tends to zero.

At most one root of (4.1) lies in S and as $\sigma \to 0$ this root tends to unity. For $\sigma \ll 1$, it may be located by expanding $\eta = 1 + i\gamma\sigma - \gamma^2\delta\sigma^2 + O(\sigma^3)$, where γ and δ are real order-one functions of a and m which are obtained by equating powers of σ. With a little algebra, one can show that, as $\sigma \to 0$, $\mathrm{sgn}\,(\mathrm{Re}\,\lambda) = \mathrm{sgn}\,(2\delta - 3)$ and

† Some details are given in appendix B.

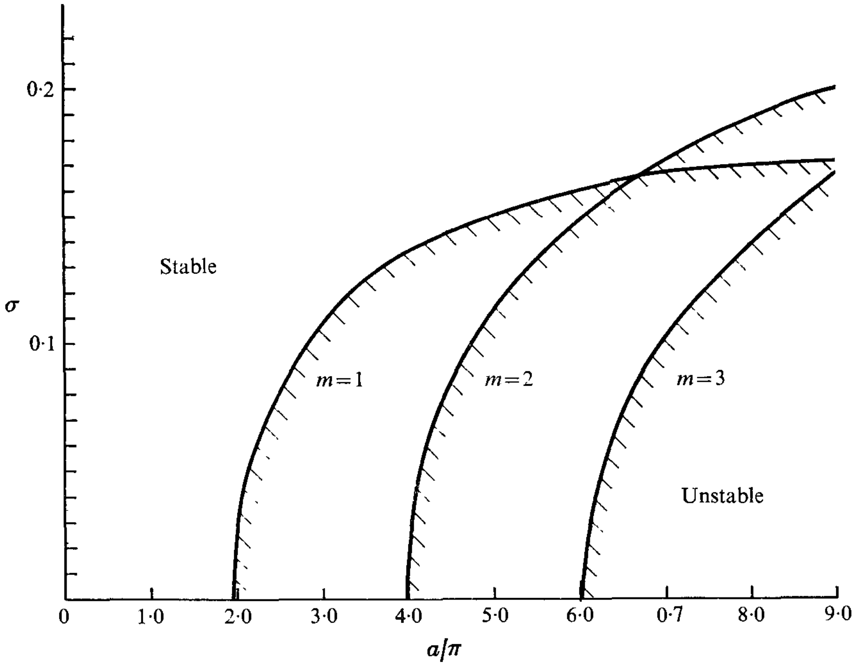

FIGURE 1. Stability diagram for the steady solutions to (2.8) showing the neutral curves for $m = 1$, 2 and 3. The unstable region for given m is on the right of the neutral curve for that m.

hence instability is assured if $\delta(a, m) > 1{\cdot}5$. The details are not given as the value of a which separates the stable and unstable regimes for given m as $\sigma \rightarrow 0$ is more easily obtained by a direct linear stability analysis of (2.8).† As outlined in appendix B, it may be shown that this value of a satisfies the equation $\phi(a, m) = 0{\cdot}5$, where the function ϕ (a positive and monotone decreasing function of a with $\phi(0, m) \equiv 1$) is defined in §4.2.

In P71, the stability analysis of (essentially) (2.1) and (2.2), with the side-wall conditions $\Phi = 0$ at $y = 0$ and $y = 1$, was performed only for small σ and instability was assured for $a^2 > 4m^2\pi^2$, compared with $a^2 > 3{\cdot}82m^2\pi^2$ in the present theory. The closeness of these results is due presumably to the fact that the constant term in the expression for $V_e(y)$, which is the only difference between the steady solution in Pedlosky's analysis and the one here (see §2), does not contribute to the stability analysis. Moreover, the side-wall energy source is negligible compared with the input from the moving horizontal boundaries in the neighbourhood of the steady solution; see §3.

In the following sections, we investigate the evolution of an incipient wave to finite amplitude in cases when the steady wave is either stable or unstable.

† Pedlosky (private communication) has also done this calculation and our results agree.

5. Solutions of the amplitude equations

5.1. *Inviscid solutions*, $\sigma = 0$

In this case, (2.8c) can be integrated to give $X_n(\tau) = \frac{1}{2}q_n(A^2 - A(0)^2)\,(n = 1, 2, \ldots)$; henceforth we omit the tilda on A. Equations (2.8a, b) may then be reduced to a single equation for $A(\tau)$:

$$d^2A/d\tau^2 = N_1 A - NA^3, \tag{5.1}$$

where
$$N = \frac{32m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}q_n, \quad N_1 = 1 + NA(0)^2. \qquad (5.2), (5.3)$$

P70 shows that (5.1) has a solution in terms of the Jacobian elliptic function dn, which describes the evolution of an incipient wave. The equation itself also has periodic solutions in terms of the elliptic function cn. Note that, in either case, the coefficient N_1 depends on the initial value of A through (5.3).

5.2. *Slightly viscous case*, $0 < \sigma \ll 1$

We begin by rewriting (2.8) in the form

$$\frac{d^2A}{d\tau^2} = -\tfrac{3}{2}\sigma\frac{dA}{d\tau} + A\left[1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_n\right] \tag{5.4}$$

and
$$d(X_n - \tfrac{1}{2}q_nA^2)/d\tau = \sigma q_n(A^2 - X_n). \tag{5.5}$$

In view of the solutions for $\sigma = 0$, we may expect that solutions of (5.4) and (5.5) for $\sigma \ll 1$ will be almost periodic, indeed almost cnoidal or dnoidal, with the amplitude and period of the oscillation changing imperceptibly on the time scale τ but significantly on the *extra slow* time scale $\theta = \sigma\tau$.† These considerations suggest the use of a multiple-scaling method and as in P72 we adopt the method developed by Kuzmak (1959) and discussed by Cole (1968). The procedure requires the introduction of a new *fast* time variable t such that $dt/d\tau = f(\theta)$, where $f(\theta)$ is an order-one function to be chosen later. The dependent variables are subsequently regarded as functions of both t and θ, which are taken to be independent. Then with

$$\frac{\partial}{\partial\tau} \equiv f\frac{\partial}{\partial t} + \sigma\frac{\partial}{\partial\theta}$$

and
$$\frac{\partial^2}{\partial\tau^2} \equiv f^2\frac{\partial^2}{\partial t^2} + \sigma\left(f'\frac{\partial}{\partial\theta} + 2f\frac{\partial^2}{\partial t\,\partial\theta}\right) + \sigma^2\frac{\partial^2}{\partial\theta^2},$$

(5.4) and (5.5) become

$$f^2\frac{\partial^2 A}{\partial t^2} - A\left[1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_n\right] = -\sigma\left[\tfrac{3}{2}f + f' + 2f\frac{\partial}{\partial\theta}\right]\frac{\partial A}{\partial t} \tag{5.6}$$

and
$$f\frac{\partial}{\partial t}(X_n - \tfrac{1}{2}q_nA^2) = \sigma\left[q_n(A^2 - X_n) - \frac{\partial}{\partial\theta}(X_n - \tfrac{1}{2}q_nA^2)\right], \tag{5.7}$$

† Note that since (2.1) and (2.2), and hence (2.8), result from an expansion of the geostrophic stream function in powers of $|\Delta|^{\frac{1}{2}}$ [see P71, equation (5.6)], their subsequent analysis in the small-σ limit, by introducing a further expansion in σ, assumes that σ is formally larger than $|\Delta|$, but smaller than $|\Delta|^{\frac{1}{2}}$.

where $f' = df/d\theta$. A solution of these equations is then sought as a power series in σ, i.e. we set

$$\begin{pmatrix} A \\ X_n \end{pmatrix} = \begin{pmatrix} A_0 \\ X_{n0} \end{pmatrix} + \sigma \begin{pmatrix} A_1 \\ X_{n1} \end{pmatrix} + O(\sigma^2) \tag{5.8}$$

in (5.6) and (5.7) and equate coefficients of powers of σ to zero.

To zeroth order in σ, we obtain

$$f^2 \frac{\partial^2 A_0}{\partial t^2} - A_0 \left[1 - \frac{64m^2}{\pi^2} \sum_{n=1}^{\infty} c_{mn} X_n \right] = 0 \tag{5.9}$$

and
$$f \partial (X_{n0} - \tfrac{1}{2} q_n A_0^2)/\partial t = 0, \tag{5.10}$$
and to first order in σ,

$$f^2 \frac{\partial^2 A_1}{\partial t^2} - A_1 \left[1 - \frac{64m^2}{\pi^2} \sum_{n=1}^{\infty} c_{mn} X_{n0} \right] + \frac{64m^2}{\pi^2} \sum_{n=1}^{\infty} c_{mn} X_{n1} A_0 = - \left[2f \frac{\partial}{\partial \theta} + f' + \tfrac{3}{2} f \right] \left(\frac{\partial A_0}{\partial t} \right) \tag{5.11}$$

and
$$f \partial (X_{n1} - q_n A_0 A_1)/\partial t = q_n (A_0^2 - X_{n0}) - \partial (X_{n0} - \tfrac{1}{2} q_n A_0^2)/\partial \theta. \tag{5.12}$$

Equation (5.10) may be integrated immediately to give

$$X_{n0} - \tfrac{1}{2} q_n A_0^2 = \Lambda_n(\theta), \tag{5.13}$$

where the $\Lambda_n(\theta)$ $(n = 1, 2, \ldots)$ are as yet undetermined functions of integration. Elimination of X_{n0} between (5.9) and (5.13) yields

$$f^2 \partial^2 A_0/\partial t^2 - N_1(\theta) A_0 + N A_0^3 = 0, \tag{5.14}$$

where
$$N_1(\theta) = 1 - \frac{64m^2}{\pi^2} \sum_{n=1}^{\infty} c_{mn} \Lambda_n(\theta). \tag{5.15}$$

Equation (5.14) is identical in form to the equation (5.1) for the inviscid problem but the coefficient $N_1(\theta)$ is a function of the slow time as given by (5.15).

Multiplication of (5.14) by $\partial A_0/\partial t$ permits a first integration of this equation, giving

$$\tfrac{1}{2} f^2 (\partial A_0/\partial t)^2 - \tfrac{1}{2} N_1(\theta) A_0^2 + \tfrac{1}{4} N A_0^4 = \mathscr{E}(\theta), \tag{5.16}$$

where $\mathscr{E}(\theta)$ is a constant of integration on the t scale. This equation for the zeroth-order amplitude has periodic solutions in terms of Jacobian elliptic functions as follows. If $\mathscr{E}(\theta) < 0$,

$$A_0(t, \theta) = \alpha(\theta) \, \mathrm{dn} \, (\omega(\theta) t, \kappa(\theta)), \tag{5.17}$$

where the frequency of oscillation $\omega(\theta) = (\tfrac{1}{2} N)^{\frac{1}{2}} \alpha/f$, the amplitude $\alpha(\theta)$ and the modulus $\kappa(\theta)$ are related to $\mathscr{E}(\theta)$ and $N_1(\theta)$ by the formulae

$$N_1(\theta) = N\alpha^2(1 - \tfrac{1}{2}\kappa^2), \quad \mathscr{E}(\theta) = \tfrac{1}{4} N \alpha^4 (\kappa^2 - 1). \tag{5.18a, b}$$

If $\mathscr{E}(\theta) > 0$,

$$A_0(t, \theta) = \alpha(\theta) \, \mathrm{cn} \, (\omega(\theta) t, \kappa(\theta)), \tag{5.19}$$

where $\omega(\theta) = (\tfrac{1}{2} N)^{\frac{1}{2}} \alpha/\kappa f$ and in this case

$$N_1(\theta) = N\alpha^4(1 - \tfrac{1}{2}/\kappa^2), \quad \mathscr{E}(\theta) = \tfrac{1}{4} N \alpha^4 (1 - \kappa^2)/\kappa^2. \tag{5.20a, b}$$

Since in the parameter range we are considering viscous effects are manifest only on the time scale σ^{-1}, the inviscid solution is appropriate when $\theta = 0$ and, as shown in P70, an initially incipient wave corresponds to $\mathscr{E}(\theta)$ slightly less than zero. Then as in §5.1 we have $\Lambda_n(0) = -\frac{1}{2}q_n\alpha(0)^2$ and $N_1(0) = 1+N\alpha(0)^2$ and without loss of generality we can take $f(0) = 1$.

To complete the zeroth-order solution we must determine the long-term drift in the functions $\alpha(\theta)$, $\kappa(\theta)$, $\Lambda_n(\theta)$, etc., as viscous effects are progressively felt on the longer time scale. In particular, we seek to answer the following question: is an exactly periodic solution of the zeroth-order equation in which α, κ, Λ_n etc. are stationary (i.e. $d\alpha/d\theta = 0$, etc. ...) possible? For $\sigma \ll 1$, this type of solution would correspond to a limit-cycle solution of (2.8). If such a solution is possible, we wish to know also if it is stable with respect to small perturbations in α, κ, Λ_n, etc., and if so, under what circumstances an incipient wave evolves to exhibit this type of behaviour.

As is common in multiple-scaling methods, equations governing changes in the parameters (e.g. $\alpha(\theta)$, $\kappa(\theta)$ and $\Lambda_n(\theta)$) of the zeroth-order solution are obtained as conditions which suppress secular terms occurring in the first-order solution. In the present problem, it is necessary to construct solutions of (5.11) and (5.12) for $A_1(t,\theta)$ and $X_{1n}(t,\theta)$ which are periodic in t with the same period θ_p as A_0 and X_{0n} such that θ_p is *independent* of θ (it is to facilitate this choice for θ_p that it is necessary to introduce the function $f(\theta)$: for details, see Cole 1968). The procedure is as follows.

For any quantity $\xi(t, \theta)$, let

$$\langle\xi\rangle = \frac{1}{\theta_p}\int_0^{\theta_p} \xi\, dt.$$

It follows that for any periodic quantity ξ with period θ_p

$$\langle d\xi/dt\rangle = 0. \tag{5.21}$$

This result applied to (5.12) with X_{n0} eliminated using (4.13) gives

$$d\Lambda_n/d\theta + q_n\Lambda_n = q_n(1-\tfrac{1}{2}q_n)\langle A_0^2\rangle. \tag{5.22}$$

After a moderate amount of manipulation, the details of which are relegated to appendix C, we deduce a further equation relating $d\alpha(\theta)/d\theta$ and $d\kappa(\theta)/d\theta$ from (5.11), namely

$$f^2\left(\frac{d}{d\theta}+\frac{f'}{f}+\tfrac{3}{2}\right)\left\langle\left(\frac{\partial A_0}{\partial t}\right)^2\right\rangle + (N_2-N)(\langle A_0^4\rangle - \langle A_0^2\rangle^2) = 0, \tag{5.23}$$

where

$$N_2 = \frac{16m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}q_n^2. \tag{5.24}$$

A final equation relating $\alpha(\theta)$, $\kappa(\theta)$ and $\Lambda_n(\theta)$ is obtained from (5.15) and either (5.18*a*) or (5.20*a*). This becomes

$$N\alpha^3(1-\tfrac{1}{2}\kappa^{-2}) = 1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}\Lambda_n \quad \text{if} \quad \mathscr{E}(\theta) > 0, \tag{5.25a}$$

$$N\alpha^2(1-\tfrac{1}{2}\kappa^2) = 1 - \frac{64m^2}{\pi^2}\sum_{r=1}^{\infty} c_{mn}\Lambda_n \quad \text{if} \quad \mathscr{E}(\theta) < 0. \tag{5.25b}$$

Finally, since t and τ are only related differentially, we may choose $\theta_p \equiv 1$ without loss of generality, implying that

$$f^2 = \begin{cases} N\alpha^2/32\kappa^2 K(\kappa)^2 & \text{if} \quad \mathscr{E}(\theta) > 0, \qquad (5.26a) \\ N\alpha^2/8K(\kappa)^2 & \text{if} \quad \mathscr{E}(\theta) < 0, \qquad (5.26b) \end{cases}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind; see, for example, Byrd & Friedman (1971, henceforth referred to as EF).

In the case $\mathscr{E}(\theta) > 0$, (5.22), (5.23), the θ derivative of (5.25a) and the logarithmic derivative of (5.26a), which gives f'/f in (5.23), lead to the following system of equations for $\alpha(\theta)$, $\kappa(\theta)$ and $\Lambda_n(\theta)$:

$$\frac{a_1(\kappa)}{\alpha}\frac{d\alpha}{d\theta} + a_2(\kappa)\frac{d\kappa}{d\theta} + a_3(\kappa, \phi) = 0, \qquad (5.27a)$$

$$a_4(\kappa, \alpha)\frac{d\alpha}{d\theta} + a_5(\kappa, \alpha)\frac{d\kappa}{d\theta} + \frac{64m^2}{\pi^2 N}\sum_{n=1}^{\infty} c_{mn}\frac{d\Lambda_n}{d\theta} = 0, \qquad (5.27b)$$

$$d\Lambda_n/d\theta + q_n\Lambda_n = q_n(1 - \tfrac{1}{2}q_n)\,b(\alpha, \kappa) \quad (n = 1, 2, \ldots), \qquad (5.27c)$$

where

$$\begin{aligned} a_1(\kappa) &= 3[(2\kappa^2 - 1)\langle \mathrm{cn}^2\rangle - \kappa^2\langle \mathrm{cn}^4\rangle + 1 - \kappa^2], \\ a_2(\kappa) &= \tfrac{1}{3}[(K'/K - \kappa^{-1})\,a_1(\kappa) + a_1'(\kappa)], \\ a_3(\kappa, \phi) &= \tfrac{1}{2}a_1(\kappa) + \kappa^2(\phi - 2)\,(\langle \mathrm{cn}^4\rangle - \langle \mathrm{cn}^2\rangle^2), \\ a_4(\alpha, \kappa) &= \alpha(2 - \kappa^{-2}), \quad a_5(\alpha, \kappa) = \alpha^2\kappa^{-3}, \\ b(\alpha, \kappa) &= \alpha^2\langle \mathrm{cn}^2\rangle, \quad \phi \equiv \phi(a, m) = 2N_2/N \end{aligned}$$

and

$$\langle \mathrm{cn}^2\rangle = (E(\kappa)/K(\kappa) - 1 + \kappa^2)\,\kappa^{-2} \quad \text{(using EF, p. 193, §312.02)},$$

$$\langle \mathrm{cn}^4\rangle = \tfrac{1}{3}[(2 - 3\kappa^2)(1 - \kappa^2) + 2(2\kappa^2 - 1)E(\kappa)/K(\kappa)]\,\kappa^{-4} \quad \text{(using EF, p. 193, §312.04)},$$

where $E(\kappa)$ is the complete elliptic integral of the second kind (EF, p. 10, §110.07). Here a prime denotes differentiation with respect to κ.

In the case $\mathscr{E}(\theta) < 0$, (5.27a–c) have the same form although the coefficients are different, namely

$$\begin{aligned} a_1(\kappa) &= 3[(2 - \kappa^2)\langle \mathrm{dn}^2\rangle - \langle \mathrm{dn}^4\rangle + \kappa^2 - 1], \\ a_2(\kappa) &= \tfrac{1}{3}[a_1(\kappa)\,K'(\kappa)/K(\kappa) + a_1'(\kappa)], \\ a_3(\kappa, \phi) &= \tfrac{1}{2}a_1(\kappa) + (\phi - 2)\,(\langle \mathrm{dn}^4\rangle - \langle \mathrm{dn}^2\rangle^2), \\ a_4(\alpha, \kappa) &= 2\alpha(1 - \tfrac{1}{2}\kappa^2), \quad a_5(\alpha, \kappa) = -\alpha^2\kappa, \\ b(\alpha, \kappa) &= \alpha^2\langle \mathrm{dn}^2\rangle, \end{aligned}$$

where $\langle \mathrm{dn}^2\rangle = E(\kappa)/K(\kappa)$ (using EF, p. 194, §314.02),

$$\langle \mathrm{dn}^4\rangle = \tfrac{1}{3}[2(2 - \kappa^2)E(\kappa)/K(\kappa) - 1 + \kappa^2] \quad \text{(using EF, p. 194, §314.04)}.$$

The system of equations (5.27) may be integrated numerically using a straightforward Runge–Kutta method if we are prepared to truncate the series occurring in (5.27b). Since $c_{mr} \sim (2r-1)^4$ for large r, the series is rapidly convergent, and in the calculations presented here, fifteen terms were taken and found to give

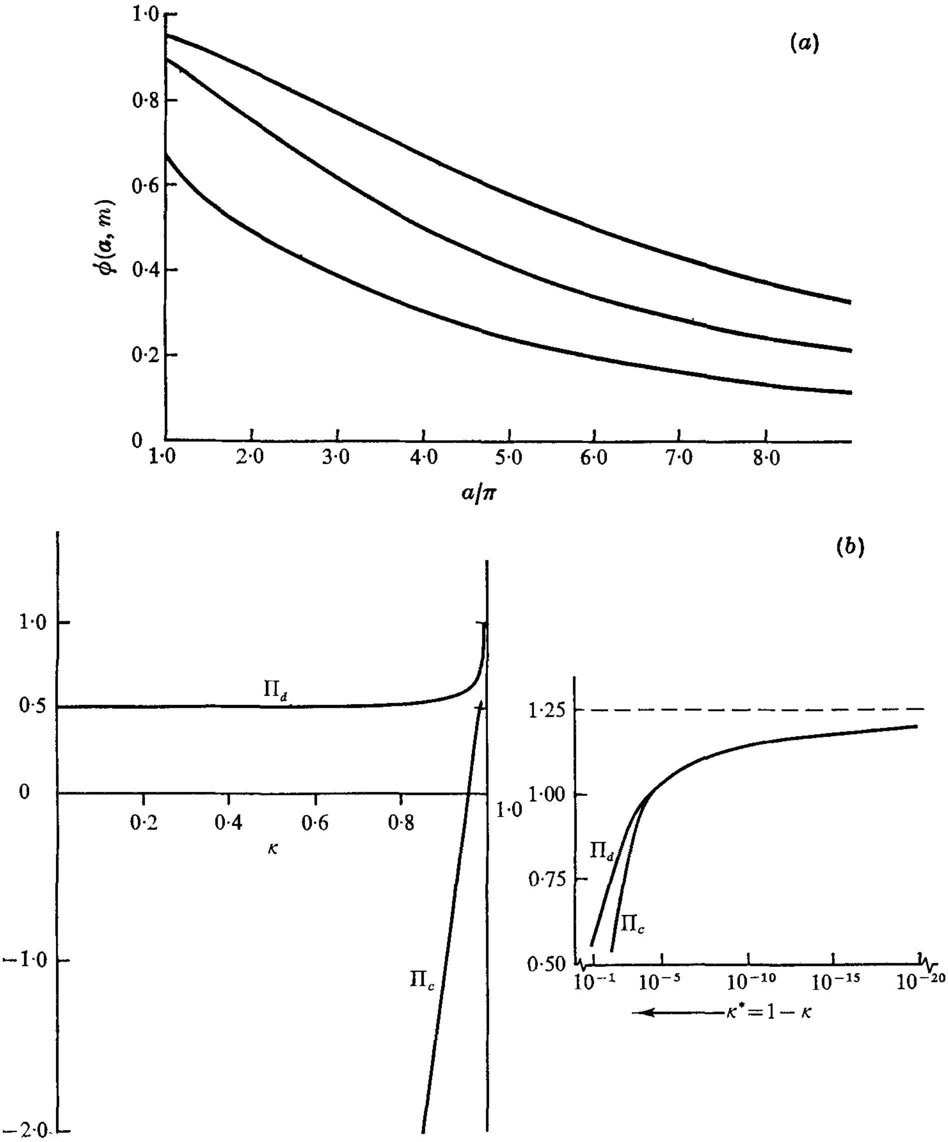

FIGURE 2. (*a*) Variation of the function $\phi(a, m)$ for selected values of m. (*b*) Variation of the functions $\Pi_c(\kappa)$ and $\Pi_d(\kappa)$: since the asymptotics of $K(\kappa)$ as $\kappa \to 1$ involve $\log(1-\kappa^2)^{\frac{1}{2}}$, the asymptotic values $\Pi_c(1) = 1{\cdot}25$, $\Pi_d(1) = 1{\cdot}25$ are approached exceedingly slowly as $\kappa \to 1$. The diagram on the right-hand side depicts the approach of the Π functions to their asymptotes, the abscissa having a logarithmic scale in $\kappa^* = 1-\kappa$.

good accuracy. In these calculations, the values of $E(\kappa)$ and $K(\kappa)$ were evaluated at each stage of the integration using standard routines. The solutions are discussed in §5.5.

5.3. *Periodic (limit-cycle) solutions*

As explained earlier, a limit-cycle solution of (2.8) corresponds, in the slightly viscous case, to a steady solution of (5.27). A necessary condition for such a

	$\mathscr{E} > 0$		$\mathscr{E} < 0$	
a/π	κ_e	α_e	κ_e	α_e
1·0	0·99445	1·71620	0·97655	1·56315
1·5	0·99072	1·72899	0·90425	1·37873
1·7	0·98957	1·73287	0·83744	1·28272
1·9	0·98852	1·73568	0·64494	1·13184
1·95	0·98826	1·73622	0·38149	1·03916
2·0	0·98801	1·73669		
2·5	0·98559	1·73856	Solutions not possible	
3·0	0·98329	1·73698		
5·0	0·97619	1·72433		

TABLE 1. Values of κ_e and α_e in limit-cycle solutions for $m = 1$ and selected values of a (in multiples of π).

solution is that there exists a value of κ, say κ_e, satisfying

$$a_3(\kappa, \phi) = 0,$$

or
$$\phi(a,m) = \begin{cases} \Pi_c(\kappa) = 2 - \frac{1}{2}a_1(\kappa)/\kappa^2(\langle \mathrm{cn}^4 \rangle - \langle \mathrm{cn}^2 \rangle^2) & \text{if } \mathscr{E} > 0, \\ \Pi_d(\kappa) = 2 - \frac{1}{2}a_1(\kappa)/(\langle \mathrm{dn}^4 \rangle - \langle \mathrm{dn}^2 \rangle^2) & \text{if } \mathscr{E} < 0. \end{cases} \tag{5.28}$$

The right-hand sides of these equations are functions of κ only whereas the left-hand side depends only on a and m. Also $a \geqslant \pi$ and $\phi(a, m)$ takes on values between $\phi(\pi, m)$ and zero, where $\phi(\pi, m)$ is bounded above by unity; see figure 2 (*a*). Since $0 < \kappa < 1$, we can readily ascertain those values of κ, if any, for which the functions Π_c and Π_d take values in the interval $[0, \phi(\pi, m)]$; see figure 2(*b*). Given a and m, accurate values for κ_e, once located, can be obtained by Newton's method. With κ_e thus determined, corresponding values for Λ_r and α, say Λ_{re} and α_e, follow from (5.27*c*) and (5.25) respectively.

Table 1 lists values of κ_e and α_e for selected values of a in the case $m = 1$. Note that, since $\Pi_d(\kappa) \geqslant \frac{1}{2}$ (figure 2*b*), the $\mathscr{E} < 0$, or dn, limit cycle is possible only when the steady wave solution (corresponding to $\alpha = 1$, $\kappa = 0$, $\Lambda_r = 1 - \frac{1}{2}q_r$, $N_1 = N$ and $E = -\frac{1}{4}N$) is stable, since then $\phi(a, m) \geqslant \frac{1}{2}$; see §4. However, the $\mathscr{E} > 0$, or cn, limit cycle is possible for all values of a. These results are remarkably similar to those of P72; in the case $\mathscr{E} < 0$, the range of variation of κ_e as a function of a is the whole interval (0, 1), whereas for $\mathscr{E} > 0$, the possible values of κ lie between 0·96373 and 0·99445 compared with 0·92877 and 0·99589 in P72. Nevertheless, it is worth re-emphasizing that the physical interpretation of the present solutions differs considerably from those of P72, unless $a/2m\pi \gg 1$ (see §3).

Of course, whether or not a limit cycle is realizable depends on its stability, which is now considered.

5.4. *Stability of the limit cycles*

The stability characteristics of the limit cycles are revealed by a *linear* stability analysis of (5.27) near the point (α_e, κ_e, Λ_{1e}, Λ_{2e}, ...): the details are straightforward and are sketched only briefly. With

$$(\alpha, \kappa, \Lambda_n) = (\alpha_e, \kappa_e, \Lambda_{ne}) + (\alpha_0, \kappa_0, \Lambda_{n0}) \exp(\lambda T),$$

the values of λ for which the linearized versions of (5.27) have a solution satisfy the transcendental equation

$$\left(\frac{\lambda(a_1 a_5/\alpha - a_2 a_4) - a_3' a_4}{\lambda(a_1 b_\kappa/\alpha - a_2 b_\alpha) - a_3' b_\alpha}\right)_e + \frac{32m^2}{\pi^2 N}\sum_{n=1}^{\infty} \frac{c_{mn} q_n (2-q_n)}{\lambda + q_n} = 0,$$

where $a_3' = \partial a_3/\partial\kappa$, $b_\alpha = \partial b/\partial\alpha$, $b_\kappa = \partial b/\partial\kappa$ and the suffix e means that a particular quantity is evaluated at the equilibrium point. Using the formula (A 8) obtained for the above series in appendix A, the equation is conveniently expressed as one in μ, where $\lambda = \mu^2/(1-\mu^2)$, i.e.

$$\left(\frac{\mu^2(a_1 a_5/\alpha - a_2 a_4 + a_3' a_4) - a_3' a_4}{\mu^2(a_1 b_\kappa/\alpha - a_2 b_\alpha + a_3' b_\alpha) - a_3' b_\alpha}\right)_e + \frac{2m^2\pi^2(2-\mu^2)}{4m^2\pi^2 + a^2\mu^2}\left[1 - \frac{4a\mu \tanh \frac{1}{2}a\mu}{4m^2\pi^2 + a^2}\right] - N = 0. \tag{5.29}$$

A limit cycle is stable unless there is at least one root of (5.29) giving $\operatorname{Re}\lambda > 0$, in which case it is unstable. The analysis follows exactly the same procedure as that used to investigate the stability of the equilibrium solution; see §3. The results are as follows: in the case of the $\mathscr{E} < 0$ limit cycle, (5.29) has a real root μ in the interval (0,1) for each value of a and this limit cycle is always unstable; however, it appears that the $\mathscr{E} > 0$ limit cycle is always stable, and this is confirmed for values of a between π and 10π in our calculations. Again, these results are exactly analogous to the ones given in P72 although the analytic details and physical interpretation are somewhat different, unless $a/2m\pi \gg 1$.

5.5. *Evolution to a limit cycle*

Equations (5.27) constitute an autonomous system in an infinite-dimensional phase space and the usual theorems available for behaviour in finite- (mostly two-) dimensional phase space are inapplicable. Nevertheless, the results for two-dimensional systems do provide some guidance in understanding the types of behaviour possible in the present problem. As the linear stability analyses of the steady solution and of the limit cycles give an indication of local features of the solutions only, we have integrated the system of equations (5.27) numerically (truncating the rapidly convergent series in (5.27 *b*) after the fifteenth term) in an attempt to gain insight into their global behaviour. The results are much as would be expected had the system been two-dimensional. *If the steady solution is stable*, it has a certain range of attraction, the 'boundary' of which may, in a loose sense, be thought of as the unstable dn limit cycle. Outside this range of attraction, the system evolves to the stable cn limit cycle. *If the steady solution is unstable*, evolution to the stable cn limit cycle occurs.

Using asymptotic formulae for the behaviour of $K(\kappa)$ and $E(\kappa)$ as $\kappa \to 1$ (see EF, p. 299, §§900.06 and 900.10) it can be shown that the coefficients a_1, a_2 and a_3 in (5.27 *a*) all tend to zero as $\kappa \to 1$. It follows that the hypersurface corresponding to $\kappa = 1$ (hence $\mathscr{E} = 0$) is a sort of 'hyperseperatrix' through which evolution from a dn solution ($\mathscr{E} < 0$) to a cn solution ($\mathscr{E} > 0$) must occur. Moreover, it appears from the numerical integrations that evolution of solutions in which the initial values of Λ_n are compatible with those of α and κ through the formula $\Lambda_n(0) = -\frac{1}{2}q_n \alpha(0)^2$, as in the case of, for example, an initially infinitesimal disturbance, or $\Lambda_n = (1-\frac{1}{2}q_n)\,b(\alpha_e, \kappa_e)$, in the case of a limit cycle, can only take place in the direction from $\mathscr{E} < 0$ to $\mathscr{E} > 0$ solutions. In other words, solutions with $\mathscr{E}$ initially positive appear always to evolve to the $\mathscr{E} > 0$ limit cycle with $\mathscr{E}$ remaining positive at all times.

6. Conclusion

The analysis described here shows that, in the case of asymptotically small (but non-zero) dissipative effects, the amplitude evolution equations for marginally unstable baroclinic waves in a two-layer, quasi-geostrophic flow model admit stable limit-cycle solutions. It resolves the uncertainty in P72 resulting from the omission of side-wall boundary conditions on the zonal mean motion but confirms the essential predictions of that paper, despite the fact that the energy conversions and mean flow structure in the present theory are, in general, substantially different to those of P72. The theory therefore offers a possible explanation for the phenomenon of amplitude vacillation in this viscous regime.

When dissipative effects are small on the time scale for the growth of an incipient wave, but not asymptotically small, approximate numerical integrations of the amplitude equations also yield periodic solutions under certain conditions. Solutions for this parameter regime will be discussed in a further paper.

I am grateful to Prof. J. Pedlosky and Dr J. Hart for helpful correspondence concerning this problem and to Prof. Steve Davis of The Johns Hopkins University for his advice concerning the presentation of certain results.

Appendix A. Evaluation of the series

$$\Sigma\, c_{mn} q_n^s (s = 0, 1, 2), \quad \Sigma\, c_{mn} q_n (2-q_n)/(\lambda + q_n)$$

The function $\sin 2m\pi x$ has the following half-range Fourier cosine series expansion over the interval [0, 1]:

$$\sin 2m\pi x = \frac{8m}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{4m^2-(2n-1)^2}. \tag{A 1}$$

The Parseval formula associated with this series is

$$\sum_{n=1}^{\infty} c_{mn} = \sum_{n=1}^{\infty} \frac{1}{[4m^2-(2n-1)^2]^2} = \frac{\pi^2}{64m^2}. \tag{A 2}$$

Since $|\sin 2m\pi x| = \sin 2m\pi x$ when $x = -1, 0$ and 1, the series (A 1) is uniformly convergent in the interval [0, 1], whence

$$\sum_{n=1}^{\infty} \frac{1}{4m^2-(2n-1)^2} \equiv 0. \tag{A 3}$$

The even function $|\sinh ax|$, $-1 \leqslant x \leqslant 1$, has the Fourier cosine expansion

$$\sinh ax = (\cosh a - 1)\left[1 + 2a \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{a^2+(2n\pi)^2}\right] - 2a(1+\cosh a) \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{a^2+(2n-1)^2\pi^2},$$

which converges uniformly in the interval $[-1, 1]$. Setting $x = 0$ and $x = 1$ and subtracting the two results leads to the formula

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\pi^2 + a^2} = \frac{1}{4a}\tanh \tfrac{1}{2}a, \tag{A 4}$$

and differentiation of each side of this equation with respect to a leads to the formula

$$\sum_{n=1}^{\infty} p_n^2 = \frac{1}{16a^2}[2\tanh \tfrac{1}{2}a - a\,\mathrm{sech}^2\,\tfrac{1}{2}a]. \tag{A 5}$$

Expressions for $N = \Sigma c_{mn} q_n$ and $N_2 = \Sigma c_{mn} q_n^2$ may then be obtained by expressing $c_{mn} q_n$ and $c_{mn} q_n^2$ as linear combinations of $c_{mn}^{\frac{1}{2}}$, c_{mn}, p_n and p_n^2. The formulae are

$$N = \frac{2m^2\pi^2}{a^2+4m^2\pi^2}\left[1 - \frac{4a\tanh\frac{1}{2}a}{a^2+4m^2\pi^2}\right] \tag{A 6}$$

and

$$N_2 = \frac{16m^2\pi^2}{(a^2+4m^2\pi^2)}\left[\tfrac{1}{4}m^2\pi^2 - \frac{2m^2\pi^2}{a^2+4m^2\pi^2}a\tanh\tfrac{1}{2}a + \tfrac{1}{8}a\left(\tanh\tfrac{1}{2}a - \frac{a}{1+\cosh a}\right)\right]. \tag{A 7}$$

Finally,

$$\frac{c_{mn}q_n(2-q_n)}{\lambda+q_n} = (2+\lambda)\frac{c_{mn}q_n}{\lambda+q_n} - c_{mn}q_n = (2-\mu^2)\frac{c_{mn}(2n-1)^2\pi^2}{(2n-1)^2\pi^2+a^2\mu^2} - c_{mn}q_n,$$

where $\lambda = \mu^2/(1-\mu^2)$. Thus, using (A 6) with a replaced by $a\mu$, it readily follows that

$$\frac{32m^2}{\pi^2}\sum_{n=1}^{\infty}\frac{c_{mn}q_n(2-q_n)}{\lambda+q_n} = \frac{2m^2\pi^2(2-\mu^2)}{4m^2\pi^2+a^2\mu^2}\left[1 - \frac{4a\mu}{4m^2\pi^2+a^2\mu^2}\tanh\tfrac{1}{2}a\mu\right] - N. \tag{A 8}$$

Note that this series is uniformly convergent for $\mathrm{Re}\,\mu > 0$.

Appendix B. Stability details for §4

With $\bar{A} = \pm 1 + \alpha$ and $X_n = 1 + x_n$, the linearized version of (2.5) gives

$$\frac{d^2\alpha}{d\tau^2} + \tfrac{3}{2}\sigma\frac{d\alpha}{d\tau} \pm \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}x_n = 0 \tag{B 1}$$

and

$$\frac{dx_n}{d\tau} + \sigma q_n x_n = \pm q_n\left[\frac{d\alpha}{d\tau} + 2a\sigma\right]. \tag{B 2}$$

It follows readily that a non-trivial solution of these equations exists, with α and x_n proportional to $\exp(\lambda\tau)$, if λ satisfies the equation

$$\lambda^2 + \tfrac{3}{2}\sigma\lambda + 2N + \frac{64m^2\sigma}{\pi^2}\sum_{n=1}^{\infty}\frac{c_{mn}q_n(2-q_n)}{\lambda+\sigma q_n} = 0. \tag{B 3}$$

When $\sigma = 0$, this equation becomes $\lambda^2 + 2N = 0$, implying imaginary values for

λ, and the points $\tilde{A} = \pm 1$, $X_n \equiv 1$ correspond to a 'centre', consistent with solutions of the inviscid equation (5.1). If $0 < \sigma \ll 1$, it is easy to show that to order σ the roots are $\lambda = \pm(-2N)^{\frac{1}{2}} + \frac{1}{4}\sigma\{1-2\phi(a,m)\}$, where $\phi(a,m) = 2N_2/N$. Hence the steady solutions $\tilde{A} = \pm 1$, $X_n \equiv 1$ are unstable if $\phi(a,m) < \frac{1}{2}$.

As indicated in §4, a similar stability analysis of the steady solutions can be done starting instead with (2.5) and (2.6). The linearized versions of these equations yield respectively

$$(\lambda^2 + \tfrac{3}{2}\sigma\lambda)\,\alpha_0 \mp 4m^2\pi^2 \int_0^1 v_0(y)\cos 2m\pi y\,dy = 0 \tag{B 4}$$

and
$$\left.\begin{aligned} &\frac{d^2v_0}{dy^2} - \frac{\lambda a^2}{\lambda+\sigma}v_0 = \pm 2\left(\frac{\lambda+2\sigma}{\lambda+\sigma}\right)\alpha_0\cos 2m\pi y,\\ &v_0(0) = 0, \quad v_0(1) = 0. \end{aligned}\right\} \tag{B 5}$$
subject to

Solving (B 5) for $v_0(y)$ and substituting this in (B 4) yields, after a little algebra,

$$\lambda^2 + \tfrac{3}{2}\sigma\lambda + \frac{4m^2\pi^2(\lambda+2\sigma)}{(4m^2\pi^2+a^2\eta^2)(\lambda+\sigma)}\left[1 - \frac{4a\eta}{4m^2\pi^2+a^2\eta^2}\tanh\tfrac{1}{2}a\eta\right] = 0, \tag{B 6}$$

where $\eta^2 = \lambda/(\lambda+\sigma)$. Written in terms of η, (B 6) is simply the discriminant equation (4.1).

Appendix C. Derivation of (5.23)

With the substitution $A_1(t,\theta) = g(t,\theta)\,\partial A_0/\partial t$, (5.11) becomes

$$f^2\left(\frac{\partial^2 g}{\partial t^2}\frac{\partial A_0}{\partial t} + \frac{\partial g}{\partial t}\frac{\partial^2 A_0}{\partial t^2} + g\frac{\partial^3 A_0}{\partial t^3}\right) - g\frac{\partial A_0}{\partial t}\left[1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_{n0}\right] + \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_{n1}A_0 = -\left(2\frac{\partial}{\partial\theta} + f' + \tfrac{3}{2}f\right)\left(\frac{\partial A_0}{\partial t}\right). \tag{C 1}$$

This may be written in the form

$$f^2\left(\frac{\partial^2 g}{\partial t^2}\frac{\partial A_0}{\partial t} + 2\frac{\partial g}{\partial t}\frac{\partial^2 A_0}{\partial t^2}\right) + g\frac{\partial}{\partial t}\left[f^2\frac{\partial^2 A_0}{\partial t^2} - A_0\left(1 - \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}X_{n0}\right)\right] + \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}\left(X_{n1} - g\frac{\partial X_{n0}}{\partial t}\right)A_0 = -\left(2\frac{\partial}{\partial\theta} + f' + \tfrac{3}{2}f\right)\left(\frac{\partial A_0}{\partial t}\right), \tag{C 2}$$

and the term in square brackets on the left-hand side of this equation vanishes on account of (5.9). Multiplication of (C 2) by $\partial A_0/\partial t$ yields

$$f^2\frac{\partial}{\partial t}\left[\left(\frac{\partial A_0}{\partial t}\right)^2\frac{\partial g}{\partial t}\right] + G(t,\theta) = -\left(\frac{\partial}{\partial\theta} + f' + \tfrac{3}{2}f\right)\left(\frac{\partial A_0}{\partial t}\right)^2, \tag{C 3}$$

where
$$G(t,\theta) = \frac{64m^2}{\pi^2}\sum_{n=1}^{\infty} c_{mn}(X_{n1} - q_nA_0A_1)\frac{\partial}{\partial t}(A_0^2),$$

using (4.10) and the definition of g given above. This expression may be rewritten in the form

$$G(t,\theta) = \frac{32m^2}{\pi^2}\left[\frac{\partial}{\partial t}\sum_{n=1}^{\infty} c_{mn} A_0^2 (X_{n1} - q_n A_0 A_1) \right.$$
$$\left. -\frac{1}{f} A_0^2 \sum_{n=1}^{\infty} c_{mn} q_n (1-\tfrac{1}{2}q_n)(A_0^2 - \langle A_0^2 \rangle)\right] \quad \text{(C 4)}$$

by using (5.12), (5.13) and (5.22). Finally, (5.23) follows from the average of (C 3) over the period θ_p if we take the form of $G(t,\theta)$ given in (C 4) and note that A_0 and A_1 being periodic in t with period θ_p implies that $\partial g/\partial t$ is periodic with the same period.

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